International Journal of Theoretical Physics, Vol. 34, No. 11, 1995

Conformal Symmetries in Static Spherically Symmetric Spacetimes

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Received January 28, 1995

We obtain the conformal symmetry vector in static, spherically symmetric spacetimes, in terms of functions subject to a number of integrability conditions that also place restrictions on the metric. Some conformal symmetries found previously are regained as special cases.

1. INTRODUCTION

Recently, conformal symmetries have generated considerable interest in the literature. These symmetries, in conjunction with the conventional isometries, help to provide us with a deeper insight into the spacetime geometry. In addition they assist in the generation of solutions, sometimes new solutions, to the Einstein field equations. As far as we are aware, Herrera *et al.* (1984) were the first to find such solutions to the field equations when modeling conformally invariant fluid spheres (which suffered from the defect of nonregularity). Maartens and Maharaj (1990) generated models of conformally invariant anisotropic fluid spheres which were regular at the center. For the application of conformal symmetries in cosmology, see, among others, Castejon-Amenedo and Coley (1992) and Maharaj *et al.* (1991).

A conformal Killing vector $\boldsymbol{\xi}$ is defined by the action of $\mathscr{L}_{\boldsymbol{\xi}}$ on the metric tensor field $\mathbf{g}$:

$$\mathscr{L}_{\boldsymbol{\xi}} g_{ij} = 2\psi g_{ij} \tag{1.1}$$

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0020-7748/95/1100-2285$07.50/0

where $\psi = \psi(x^i)$ is the conformal factor. In spite of extensive analyses of conformal symmetries, equation (1.1) has been integrated in general only in a few cases: Minkowski spacetimes (Choquet-Bruhat *et al.*, 1977), Robertson–Walker spacetimes (Maartens and Maharaj, 1986), and *pp*-wave spacetimes (Maartens and Maharaj, 1991). To make progress one often has to impose another restriction on the conformal vector $\boldsymbol{\xi}$. To simplify the analysis of (1.1), Herrera *et al.* (1984) assumed that

$$\mathcal{L}_{\xi} u_i = \psi u_i \tag{1.2}$$

where $\mathbf{u}$ is the fluid 4-velocity, so that fluid flow lines are mapped conformally onto fluid flow lines. The existence of a conformal Killing vector $\boldsymbol{\xi}$ does not necessarily imply (1.2); Maartens *et al.* (1986) showed that (1.2) is in fact a special case of

$$\mathcal{L}_{\xi} u_i = \psi u_i + v_i$$

where $u^i v_i = 0$, and studied the kinematical and dynamical properties of anisotropic fluids for this general case. Coley and Tupper (1990a) called vectors satisfying (1.2) *inheriting* conformal Killing vectors as fluid flow lines are mapped conformally. They subsequently investigated the existence of inheriting vectors in spherically symmetric spacetimes with a perfect-fluid energy-momentum tensor (Coley and Tupper, 1990b) and with anisotropic fluids (Coley and Tupper, 1994).

We consider a static, spherically symmetric spacetime with coordinates $(x^i) = (t, r, \theta, \phi)$ so that the line element is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{1.3}$$

Our intention is to investigate the conformal symmetries of the spacetime (1.3). We integrate (1.1) without any additional assumptions; in particular, we do not assume *a priori* the inheriting condition (1.2). Coley and Tupper (1994) make the observation that most conformal Killing vectors known are in spherically symmetric spacetimes. Thus a systematic study of conformal symmetries in spacetimes with spherical symmetry is needed, as conformal symmetries are rare. Toward this end, we explicitly determine the extra conditions that the conformal symmetry places on the metric tensor field. These conditions provide the basis for a (future) study of solutions to the Einstein field equations with the general conformal symmetry obtained here.

2. CONFORMAL EQUATIONS

In this section we analyze the conformal symmetry for static, spherically symmetric spacetimes without making any assumptions about the form of the conformal vector. The conformal Killing vector equation (1.1) for the metric (1.3) reduces to the following system of equations:

$$\nu'\xi^1 + \xi^0_t = \psi \tag{2.1a}$$

$$-e^{2\nu}\xi^0_r + e^{2\lambda}\xi^1_t = 0 \tag{2.1b}$$

$$-e^{2\nu}\xi^0_\theta + r^2\xi^2_t = 0 \tag{2.1c}$$

$$-e^{2\nu}\xi^0_\phi + r^2 \sin^2\theta \, \xi^3_t = 0 \tag{2.1d}$$

$$\lambda'\xi^1 + \xi^1_r = \psi \tag{2.1e}$$

$$e^{2\lambda}\xi^1_\theta + r^2\xi^2_r = 0 \tag{2.1f}$$

$$e^{2\lambda}\xi^1_\phi + r^2 \sin^2\theta \, \xi^3_r = 0 \tag{2.1g}$$

$$\xi^1 + r\xi^2_\theta = r\psi \tag{2.1h}$$

$$\xi^2_\phi + \sin^2\theta \, \xi^3_\theta = 0 \tag{2.1i}$$

$$\xi^1 + r \cot\theta \, \xi^2 + r\xi^3_\phi = r\psi \tag{2.1j}$$

where subscripts denote partial differentiation and primes denote differentiation with respect to r. The equations (2.1) are a coupled system of first-order, linear partial differential equations for the conformal vector $\boldsymbol{\xi} = (\xi^0, \xi^1, \xi^2, \xi^3)$ and the conformal factor ψ.

It is possible to integrate the system (2.1) in general. In a tedious, but straightforward calculation we find that the components ξ^0, ξ^1, ξ^2, ξ^3, and ψ decouple and a number of integrability conditions are generated. We do not provide details of the calculation, as the integration process is standard [see Maartens and Maharaj (1991) for the procedure] and lengthy; it is a simple matter to confirm that the given solution satisfies (2.1) by direct substitution. The general solution of the system (2.1) is given by the components of the vector $\boldsymbol{\xi}$:

$$\xi^0 = r^2e^{-2\nu} \sin\theta \, [\mathcal{A}_t \sin\phi - \mathcal{B}_t \cos\phi] - r^2e^{-2\nu}\mathcal{C}_t \cos\theta + \mathcal{J} \tag{2.2a}$$

$$\xi^1 = r^2e^{-2\lambda} \sin\theta \, [-\mathcal{A}_r \sin\phi + \mathcal{B}_r \cos\phi] + r^2e^{-2\lambda}\mathcal{C}_r \cos\theta + \mathcal{H} \tag{2.2b}$$

$$\xi^2 = \cos\theta \, [\mathcal{A} \sin\phi - \mathcal{B} \cos\phi] + \mathcal{C} \sin\theta + a_1 \sin\phi + a_2 \cos\phi \tag{2.2c}$$

$$\xi^3 = \csc\theta \, [\mathcal{A} \cos\phi + \mathcal{B} \sin\phi] - \cot\theta \, (a_1 \cos\phi + a_2 \sin\phi) + a_3 \tag{2.2d}$$

and the conformal factor

$$\begin{aligned} \psi = {} & r^2 \sin\theta \sin\phi \, (-\nu'e^{-2\lambda}\mathcal{A}_r + e^{-2\nu}\mathcal{A}_{tt}) \\ & + r^2 \sin\theta \cos\phi \, (\nu'e^{-2\lambda}\mathcal{B}_r - e^{-2\nu}\mathcal{B}_{tt}) \\ & + r^2 \cos\theta \, (\nu'e^{-2\lambda}\mathcal{C}_r - e^{-2\nu}\mathcal{C}_{tt}) + \nu'\mathcal{H} + \mathcal{J}_t \end{aligned} \tag{2.3}$$

where $\mathcal{A} = \mathcal{A}(t, r)$, $\mathcal{B} = \mathcal{B}(t, r)$, $\mathcal{C} = \mathcal{C}(t, r)$, $\mathcal{F} = \mathcal{F}(t, r)$, and $\mathcal{H} = \mathcal{H}(t, r)$ are functions of integration and a_1, a_2, and a_3 are constants. This solution is subject to the following integrability conditions:

$$e^{2\nu}(r^2e^{-2\nu}\mathcal{A}_t)_r + r^2\mathcal{A}_{tr} = 0 \tag{2.4a}$$

$$r^2\nu'e^{-2\lambda}\mathcal{A}_r - r^2e^{-2\nu}\mathcal{A}_{tt} = \lambda'r^2e^{-2\lambda}\mathcal{A}_r + (r^2e^{-2\lambda}\mathcal{A}_r)_r \tag{2.4b}$$

$$re^{-2\lambda}\mathcal{A}_r + \mathcal{A} = r^2(\nu'e^{-2\lambda}\mathcal{A}_r - e^{-2\nu}\mathcal{A}_{tt}) \tag{2.4c}$$

$$e^{2\nu}(r^2e^{-2\nu}\mathcal{B}_t)_r + r^2\mathcal{B}_{tr} = 0 \tag{2.4d}$$

$$r^2\nu'e^{-2\lambda}\mathcal{B}_r - r^2e^{-2\nu}\mathcal{B}_{tt} = \lambda'r^2e^{-2\lambda}\mathcal{B}_r + (r^2e^{-2\lambda}\mathcal{B}_r)_r \tag{2.4e}$$

$$re^{-2\lambda}\mathcal{B}_r + \mathcal{B} = r^2(\nu'e^{-2\lambda}\mathcal{B}_r - e^{-2\nu}\mathcal{B}_{tt}) \tag{2.4f}$$

$$e^{2\nu}(r^2e^{-2\nu}\mathcal{C}_t)_r + r^2\mathcal{C}_{tr} = 0 \tag{2.4g}$$

$$r^2\nu'e^{-2\lambda}\mathcal{C}_r - r^2e^{-2\nu}\mathcal{C}_{tt} = \lambda'r^2e^{-2\lambda}\mathcal{C}_r + (r^2e^{-2\lambda}\mathcal{C}_r)_r \tag{2.4h}$$

$$re^{-2\lambda}\mathcal{C}_r + \mathcal{C} = r^2(\nu'e^{-2\lambda}\mathcal{C}_r - e^{-2\nu}\mathcal{C}_{tt}) \tag{2.4i}$$

$$\left(\frac{1}{r} - \nu'\right)\mathcal{H} = \mathcal{F}_t \tag{2.4j}$$

$$e^{2\nu}\mathcal{F}_r - e^{2\lambda}\mathcal{H}_t = 0 \tag{2.4k}$$

$$\nu'\mathcal{H} + \mathcal{F}_t = \lambda'\mathcal{H} + \mathcal{H}_r \tag{2.4l}$$

We emphasize that the solution (2.2)–(2.3), subject to the integrability conditions (2.4), is the most general conformal symmetry of the static, spherically symmetric spacetime (1.3). Note that in the above solution the angular dependance (θ, ϕ) of the conformal vector ξ has been completely determined; there is freedom only in the t and r coordinates. Also, the conformal symmetry is nonstatic in general, so that the static property of the spacetime is not inherited by the conformal symmetry.

3. SOME SPECIAL CASES

First consider the special case of Killing symmetries. By setting $\psi = 0$ in (2.3) and analyzing the integrability conditions (2.4), we obtain

$$\mathcal{A} = 0, \qquad \mathcal{B} = 0, \qquad \mathcal{C} = 0, \qquad \mathcal{H} = 0, \qquad \mathcal{F} = \text{const}$$

Thus the general Killing vector of (1.3) is

$$\xi = \mathcal{J}\frac{\partial}{\partial t} + (-a_1 \sin\phi + a_2 \cos\phi)\frac{\partial}{\partial\theta}$$
$$+ [-\cot\theta\,(a_1\cos\phi + a_2\sin\phi) + a_3]\frac{\partial}{\partial\phi} \tag{3.1}$$

We generate the familiar four-dimensional Lie algebra of the Killing vectors spanned by the vectors

$$\xi_0 = \frac{\partial}{\partial t}, \qquad \xi_1 = \frac{\partial}{\partial\phi}$$
$$\xi_2 = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}$$
$$\xi_3 = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}$$

by appropriate choices of the constants a_1, a_2, a_3, and $\mathcal{J}$ in (3.1).

Spherical symmetry has motivated the choice of the conformal Killing vector

$$\xi = \xi^0(t, r)\frac{\partial}{\partial t} + \xi^1(t, r)\frac{\partial}{\partial r}$$

with a static conformal factor $\psi = \psi(r)$ in previous analyses. For consistency with this form of the conformal symmetry the functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and the constants a_1, a_2, a_3 in (2.2) vanish. The conformal factor (2.3) and the integrability conditions (2.4) may be expressed as

$$\psi = \frac{1}{r}\mathcal{K}$$
$$\left(\frac{1}{r} - \nu'\right)\mathcal{K} = \mathcal{J}_t$$
$$e^{2\nu}\mathcal{J}_r - e^{2\lambda}\mathcal{K}_t = 0$$
$$\nu'\mathcal{K} + \mathcal{J}_t = \lambda'\mathcal{K} + \mathcal{K}_r$$

We integrate this system to obtain the vector

$$\xi = (\mathcal{J}_1 t + \mathcal{J}_2)\frac{\partial}{\partial t} + r\psi\frac{\partial}{\partial r} \tag{3.2}$$

and the gravitational potentials are given by

$$e^{2\nu} = C^2 r^2 \exp\left(-2\mathcal{J}_1 B^{-1} \int r^{-1} e^{-\lambda} dr\right)$$

$$e^{2\lambda} = \frac{B^2}{\psi^2}$$

where $\mathcal{J}_1$, $\mathcal{J}_2$, B, and C are constants.

This form of the conformal Killing vector (3.2) has been used by Maartens and Maharaj (1990) to construct static conformally invariant anisotropic solutions. We regain their results if we set

$$B = 1, \qquad \mathcal{J}_1 = k, \qquad \mathcal{J}_2 = A$$

Herrera *et al.* (1984) and Herrera and Ponce de Leon (1985) considered the simpler case of a static vector $\xi = (\xi^0(r), \xi^1(r), 0, 0)$. Their case is regained if we set $\mathcal{J}_1 = k = 0$. The solutions of Herrera *et al.* and Herrera and Ponce de Leon, with a static conformal vector, are irregular at the center. The solutions of Maartens and Maharaj, with a nonstatic conformal vector but static conformal factor, are regular at the center, but have negative pressures. It may be possible to overcome this undesirable physical feature if the conformal factor is nonstatic, $\psi = \psi(t, r)$.

The solution of the conformal Killing equation presented here forms the basis for a systematic classification of conformal symmetries, including inheriting symmetries. This is a problem presently under study and we intend to publish a comprehensive treatment of conformal symmetries in static, spherically symmetric spacetimes in the future.

ACKNOWLEDGMENTS

S.D.M. thanks David Matravers and the University of Portsmouth for hospitality and support where this work was completed. The authors thank the FRD for financial support.

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